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Weighted Approximation by Szász–Mirakjan Operators*

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In this paper we consider the weighted approximation by the Szász–Mirakjan operators. We characterize the functions with nonoptimal approximation order by smoothness.

1. INTRODUCTION

The Szász–Mirakjan operators in $[0, \infty)$ are given by

$$S_n(f, x) = \sum_{k=0}^{\infty} f(k/n)\, p_{n,k}(x), \qquad p_{n,k}(x) = e^{-nx}(nx)^k/k!. \tag{1.1}$$

In 1978, M. Becker [1] proved for $m \in N \cup \{0\}$, $(1+x)^{-m} f(x) \in L_\infty[0, \infty)$ and $0 < \alpha < 2$ that

$$\begin{aligned} &(1+x)^{-m}\, |S_n(f, x) - f(x)| \leqslant M_f (x/n)^{\alpha/2} \qquad (x \geqslant 0, n \in N) \\ \Leftrightarrow\ &(1+x)^{-m}\, |f(x+2h) - 2f(x+h) + f(x)| \leqslant M_f' h^\alpha \qquad (h > 0, x \geqslant 0). \end{aligned} \tag{1.2}$$

V. Totik [7] gave a characterization theorem for these operators in 1983. He proved for $f \in C[0, \infty) \cap L_\infty[0, \infty)$ and $0 < \alpha < 2$ that

$$\begin{aligned} &\|S_n(f) - f\|_\infty = O(n^{-\alpha/2}) \\ \Leftrightarrow\ &x^{\alpha/2}\, |f(x+2h) - 2f(x+h) + f(x)| \leqslant M_f h^\alpha \qquad (x \geqslant 0, h > 0). \end{aligned} \tag{1.3}$$

This result was also proved by V. Totik [6], Z. Ditzian and V. Totik [4].

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In this paper we shall consider the weighted approximation by the Szász–Mirakjan operators and give a characterization theorem. We take spaces $C_{a,b}$ via the weights $w_{a,b}$ as follows.

$$w(x) := w_{a,b}(x) = x^a(1+x)^{-b}, \qquad 1 > a > 0, b > 0, \tag{1.4}$$

$$C_{a,b} = \{f \in C[0,\infty) : wf \in L_\infty[0,\infty)\}, \tag{1.5}$$

$$\|f\|_w = \|wf\|_\infty. \tag{1.6}$$

In their monograph in 1987 [4], Z. Ditzian and V. Totik gave some weighted approximation theorems for Kantorovich type integral version of the exponential-type operators. It is strange that they did not consider the same problems for the exponential-type operators. We shall point out that the Szász–Mirakjan operators are unbounded in $C_{a,b}$ with the norm (1.6). However, since the Szász–Mirakjan operators reproduce linear functions, we only need to discuss in the space $C^0_{a,b} = \{f \in C_{a,b} : f(0) = 0\}$ first and then extend to $C_{a,b}$. Let us denote $[x]$ as the integer part of $x > 0$.

2. An Unbounded Property

We show the unbounded property for the Szász–Mirakjan operators as follows.

Lemma 2.1. *For $S_n(f, x)$ given by (1.1) and $f \in C_{a,b}$, we have*

$$\left| w(x) \sum_{k=1}^{\infty} f(k/n)\, p_{n,k}(x) \right| \leq M_{a,b} \|f\|_w, \tag{2.1}$$

where $M_{a,b}$ is a constant depending only on a and b. This implies for $f \in C^0_{a,b}$ that

$$\|S_n(f)\|_w \leq M_{a,b} \|f\|_w. \tag{2.2}$$

Proof. Note that [1] for $m \in N$

$$S_n(1 + t^m, x) \leq M_m(1 + x^m). \tag{2.3}$$

We have

$$S_n((1+t)^m, x) \leq 2^m S_n(1 + t^m, x) \leq 2^m M_m(1+x)^m. \tag{2.4}$$

By choosing $m=[b/(1-a)]+1$ we then have

$$\left|w(x)\sum_{k=1}^{\infty} f(k/n)\,p_{n,k}(x)\right| \leqslant w(x)\left(\sum_{k=1}^{\infty}(n/k)\,p_{n,k}(x)\right)^{a}$$

$$\times\left(\sum_{k=1}^{\infty}(1+k/n)^{m}\,p_{n,k}(x)\right)^{b/m}\|f\|_{w}$$

$$\leqslant 2^{a}2^{b}(M_{m})^{b/m}\|f\|_{w}.$$

Our proof is then complete.

THEOREM 1. *For any $n\in N$, the Szász–Mirakjan operator $S_n(f,x)$ is unbounded in $(C_{a,b},\|\cdot\|_w)$.*

Proof. Let $f_m(x)=1/(x^a+1/m)$. Then we have $\|f_m\|_w\leqslant 1$. By Lemma 2.1 we have

$$\|S_n(f_m)\|_w \geqslant \|w(x)\,f_m(0)\,p_{n,0}(x)\|_\infty - \left\|w(x)\sum_{k=1}^{\infty} f_m(k/n)\,p_{n,k}(x)\right\|_\infty$$

$$\geqslant m\,\|w(x)\,e^{-nx}\|_\infty - M_{a,b}$$

$$\rightarrow\infty \qquad (m\rightarrow\infty).$$

Hence $S_n(f)$ is unbounded in $(C_{a,b},\|\cdot\|_w)$.

3. BERNSTEIN TYPE INEQUALITIES

The main tool for the proof of the inverse theorem in the nonoptimal case is an appropriate Bernstein-type inequality. Denote $\varphi(x)=x$.

LEMMA 3.1. *Let $c,d\geqslant 0$. Then we have*

$$\left|\sum_{k=1}^{\infty}(k/n)^{-c}(1+k/n)^{d}\,p_{n,k}(x)\right| \leqslant M_{c,d}\,x^{-c}(1+x)^{d}, \qquad (x>0), \tag{3.1}$$

where $M_{c,d}$ is a constant depending only on c and d.

Proof. If $c,d>0$, then we have for $x>0$

$$\left|\sum_{k=1}^{\infty}(k/n)^{-c}(1+k/n)^{d}\,p_{n,k}(x)\right|$$

$$\leqslant\left(\sum_{k=1}^{\infty}(k/n)^{-2c}\,p_{n,k}(x)\right)^{1/2}\left(\sum_{k=1}^{\infty}(1+k/n)^{2d}\,p_{n,k}(x)\right)^{1/2}$$

$$\leqslant \left(\sum_{k=1}^{\infty} (n/k)^{[2c]+1} p_{n,k}(x)\right)^{c/([2c]+1)} (S_n((1+t)^{[2d]+1}, x))^{d/([2d]+1)}$$
$$\leqslant (([2c]+2)!\, x^{-([2c]+1)})^{c/([2c]+1)}$$
$$\times (2^{[2d]+1} M_{[2d]+1} (1+x)^{[2d]+1})^{d/([2d]+1)}$$
$$\leqslant M_{c,d}\, x^{-c}(1+x)^d.$$

The cases of $c=0$ or $d=0$ can be easily obtained, and our proof is then complete.

LEMMA 3.2 (Bernstein-type Inequality). *Let $f \in C^0_{a,b}$, $n \in N$. Then we have*

$$\|\varphi S_n''(f)\|_w \leqslant M_1 n \|f\|_w, \tag{3.2}$$

where M_1 is a constant independent of f and n.

Proof. Note that for $g \in C[0,\infty)$

$$S_n''(g, x) = n^2 \sum_{k=0}^{\infty} (g((k+2)/n) - 2g((k+1)/n) + g(k/n))\, p_{n,k}(x), \tag{3.3}$$

$$S_n''(g, x) = (n/x)^2 \sum_{k=0}^{\infty} g(k/n)((k/n - x)^2 - kn^{-2})\, p_{n,k}(x). \tag{3.4}$$

Then for $x \in (0, 1/n]$ we have by Lemma 3.1 and (3.3)

$$|w(x)\, \varphi(x)\, S_n''(f, x)| \leqslant w(x)\, xn^2 \sum_{k=1}^{\infty} 4(k/n)^{-a} (1+k/n)^b\, p_{n,k}(x)\, \|f\|_w$$
$$\leqslant 4nw(x)\, M_{a,b} x^{-a}(1+x)^b \|f\|_w$$
$$\leqslant M_1 n \|f\|_w.$$

For $x > 1/n$, by (3.4) we have

$$|w(x)\, \varphi(x)\, S_n''(f, x)|$$
$$\leqslant w(x)\, n^2/x \sum_{k=1}^{\infty} (w(k/n))^{-1} ((k/n - x)^2 + kn^{-2})\, p_{n,k}(x)\, \|f\|_w$$
$$\leqslant w(x)\, n^2/x \left(\sum_{k=1}^{\infty} (k/n)^{-2a} (1+k/n)^{2b}\, p_{n,k}(x)\right)^{1/2}$$
$$\times (2S_n((t-x)^4 + t^2 n^{-2}, x))^{1/2} \|f\|_w$$
$$\leqslant \sqrt{M_{2a,2b}}\, n^2/x(2(xn^{-3} + 3x^2n^{-2} + x^2n^{-2} + xn^{-3}))^{1/2} \|f\|_w$$
$$\leqslant 4\sqrt{M_{2a,2b}}\, n \|f\|_w$$
$$\leqslant M_1 n \|f\|_w,$$

here we have used the moments of the Szász–Mirakjan operators

$$S_n((t-x)^2, x) = x/n,$$
$$S_n((t-x)^4, x) = xn^{-3} + 3x^2n^{-2}. \tag{3.5}$$

Thus we have proved our Bernstein-type inequality.

To prove our direct and inverse results we need the Peetre's K-functional defined in $C^0_{a,b}$ as

$$K(f, t)_w = \inf_{g \in D} \{\|f-g\|_w + t\,\|\varphi g''\|_w\}, \tag{3.6}$$

$$D = \{g \in C^0_{a,b} : g' \in \mathrm{A.C.}_{\mathrm{loc}},\ \|\varphi g''\|_w < \infty\}. \tag{3.7}$$

LEMMA 3.3. *Let $f \in D$, $n \in N$. Then we have*

$$\|\varphi S_n''(f)\|_w \leqslant M_2\,\|\varphi f''\|_w, \tag{3.8}$$

where M_2 is a constant independent of f and n.

Proof. Let $x > 0$, $n \in N$. By (3.3) and Lemma 3.1 we have

$$\begin{aligned}
|w(x)\,\varphi(x)\,S_n''(f, x)| &= \left| w(x)\,xn^2 \sum_{k=0}^{\infty} \iint_0^{1/n} f''(k/n+u+v)\,du\,dv\,p_{n,k}(x)\right| \\
&\leqslant w(x)\,xn^2 \sum_{k=0}^{\infty} \iint_0^{1/n} (k/n+u+v)^{-1-a} \\
&\quad \times (1+k/n+u+v)^b\,du\,dv\,p_{n,k}(x)\,\|\varphi f''\|_w \\
&\leqslant w(x)\,x \sum_{k=1}^{\infty} \{(1+(k+2)/n)^b\,(k/n)^{-1-a}\,p_{n,k}(x)\} \\
&\quad \times \|\varphi f''\|_w + w(x)\,xn^2(1+2/n)^b \\
&\quad \times \int_0^{1/n} u^{-a}/a\,du\,p_{n,0}(x)\,\|\varphi f''\|_w \\
&\leqslant w(x)\,x \sum_{k=1}^{\infty} \{3^b(1+k/n)^b\,(k/n)^{-1-a}\,p_{n,k}(x)\}\,\|\varphi f''\|_w \\
&\quad + w(x)\,xn^2 3^b(1/n)^{1-a}/(a(1-a))\,p_{n,0}(x)\,\|\varphi f''\|_w \\
&\leqslant w(x)\,x\,3^b M_{1+a,b} x^{-1-a}(1+x)^b\,\|\varphi f''\|_w \\
&\quad + 3^b(nx)^{1+a}e^{-nx}/(a(1-a))\,\|\varphi f''\|_w \\
&\leqslant M_2\,\|\varphi f''\|_w.
\end{aligned}$$

Our proof is therefore complete.

4. A Characterization Theorem

With all the above preparations we can now give our characterization theorem. Denote $\Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$ for $x \geq 0$.

THEOREM 2. *Let* $0 < a < 1$, $b > 0$, $w(x) = x^a(1+x)^{-b}$, $f \in C_{a,b}$. *Then for* $0 < \alpha < 1$, *the following statements are equivalent*:

$$(1) \quad w(x)\,|S_n(f, x) - f(x)| \leq M_f n^{-\alpha} \qquad (n \in N,\, x \geq 0). \tag{4.1}$$

$$(2) \quad K(f, t)_w \leq M_f' t^\alpha \qquad (t > 0). \tag{4.2}$$

$$(3) \quad \begin{aligned} &\sup_{x \geq 0} |x^{a+\alpha}(1+x+2h)^{-b}\, \Delta_h^2 f(x)| \leq M_f'' h^{2\alpha} \qquad (h > 0); \\ &|w(x)\, x^\alpha \Delta_x^2 f(0)| \leq M_f'' x^{2\alpha} \qquad (x > 0). \end{aligned} \tag{4.3}$$

Proof. It is sufficient to prove this theorem for $f \in C_{a,b}^0$.

By the standard method for the inverse results [4, 5] we have the implication $(1) \Rightarrow (2)$ from Lemmas 2.1, 3.2, and 3.3.

Now suppose (2) holds. We want to prove (3).

Let $x > 0$. Then we have

$$\begin{aligned}
&|x^a(1+x+2h)^{-b}\, \Delta_h^2 f(x)| \\
&\quad \leq x^a(1+x+2h)^{-b}\,(1/w(x) + 2/w(x+h) + 1/w(x+2h)) \\
&\qquad \times \|f-g\|_w + x^a(1+x+2h)^{-b} \iint_0^h |g''(x+u+v)|\, du\, dv \\
&\quad \leq 4\,\|f-g\|_w + x^a(1+x+2h)^{-b} \\
&\qquad \times \iint_0^h (x+u+v)^{-a-1}\,(1+x+u+v)^b\, du\, dv\, \|\varphi g''\|_w \\
&\quad \leq 4\{\|f-g\|_w + h^2/x\, \|\varphi g''\|_w\}.
\end{aligned}$$

By taking infimum for $g \in D$ we obtain

$$x^a(1+x+2h)^{-b}\,|\Delta_h^2 f(x)| \leq 4K(f, h^2/x)_w \leq 4M_f'(h^2/x)^\alpha.$$

Hence the first statement of (4.3) is valid. The second statement can be proved in the same way.

We now want to prove the final implication $(3) \Rightarrow (1)$.

Introducing the Steklov type means for $h > 0$ by

$$f_h(x) = (2/h)^2 \iint_0^{h/2} (2f(x+u+v) - f(x+2u+2v))\, du\, dv, \tag{4.4}$$

one has [1]

$$\begin{aligned} f(x)-f_h(x) &= (2/h)^2 \iint_0^{h/2} \Delta_{u+v}^2 f(x)\, du\, dv, \\ f_h''(x) &= h^{-2}(8\Delta_{h/2}^2 f(x) - \Delta_h^2 f(x)). \end{aligned} \tag{4.5}$$

Suppose that (4.3) holds. For $x>0$, $n\in N$, let $h=(x/n)^{1/2}$. Note that $(t-u)\,u^{-a}$ is monotone for $u\in[t,x]$ or $[x,t]$. Then we have

$$\begin{aligned} & w(x)\,|S_n(f_h,x)-f_h(x)| \\ &\quad \leqslant w(x)\, S_n\left(\int_x^t (t-u)\,|f_h''(u)|\, du, x\right) \\ &\quad \leqslant 9M_f''\, w(x)\, h^{-2} S_n\left(\int_x^t (t-u)\, h^{2\alpha} u^{-a-\alpha}(1+u+2h)^b\, du, x\right) \\ &\quad \leqslant 9M_f''\, w(x)\, h^{2\alpha-2} \\ &\qquad \times S_n\left((t-x)\, x^{-a} \int_x^t u^{-\alpha}\, du((1+x+2h)^b + (1+t+2h)^b), x\right). \end{aligned}$$

If $x\leqslant 1/n$, then we have $h\leqslant 1$ and

$$(t-x)\int_x^t u^{-\alpha}\, du = (t-x)(t^{1-\alpha}-x^{1-\alpha})/(1-\alpha) \leqslant |t-x|^{2-\alpha}/(1-\alpha).$$

Hence

$$\begin{aligned} w(x)\,|S_n(f_h,x)-f_h(x)| &\leqslant 9M_f''(1+x)^{-b}\, h^{2\alpha-2}/(1-\alpha)(S_n((t-x)^2,x))^{1-\alpha/2} \\ &\qquad \times 2(S_n((1+x+2h)^{2b/\alpha} + (1+t+2h)^{2b/\alpha}, x))^{\alpha/2} \\ &\leqslant 18M_f''(1+x)^{-b}\,(x/n)^{\alpha-1}/(1-\alpha)(x/n)^{1-\alpha/2} \\ &\qquad \times ((1+x+2h)^{2b/\alpha} + 3^{2b/\alpha} S_n((1+t)^{2b/\alpha}, x))^{\alpha/2} \\ &\leqslant 18M_f''(1+x)^{-b}/(1-\alpha)(x/n)^{\alpha/2} \\ &\qquad \times 3^b((1+x)^{2b/\alpha} + M_{a,b}(1+x)^{2b/\alpha})^{\alpha/2} \\ &\leqslant M_f n^{-\alpha}. \end{aligned}$$

Let

$$S_n^*(g,y) = \sum_{k=1}^{\infty} g(k/n)\, p_{n,k}(y). \tag{4.6}$$

If $x > 1/n$, then we have

$$w(x)\,|S_n(f_h,x)-f_h(x)| \leqslant 9M_f'' h^{2\alpha-2}(1+x)^{-b} \times (S_n(((1+x+2h)^b+(1+t+2h)^b)^2,x))^{1/2} \times \left(S_n\left(\left((t-x)\int_x^t u^{-\alpha}\,du\right)^2,x\right)\right)^{1/2} \tag{4.7}$$

Note that $1+x+2h \leqslant 3(1+x)$ and

$$(1+t+2h)^{2b} \leqslant 2^{2b}((1+t)^{2b}+(2h)^{2b}).$$

We have

$$\begin{aligned}(S_n(((1+x+2h)^b+(1+t+2h)^b)^2,x))^{1/2} &\leqslant (2((1+x+2h)^{2b}+2^{2b}S_n((2h)^{2b}+(1+t)^{2b},x)))^{1/2}\\ &\leqslant M_b(1+x)^b.\end{aligned}$$

By the moments of the Szász–Mirakjan operators [4] we have

$$\begin{aligned}S_n\left(\left((t-x)\int_x^t u^{-\alpha}\,du\right)^2,x\right) &\leqslant S_n^*(((t-x)^2\,(x^{-\alpha}+t^{-\alpha}))^2,x)\\ &\quad+\left(x\int_0^x u^{-\alpha}\,du\right)^2 p_{n,0}(x)\\ &\leqslant (S_n((t-x)^8,x))^{1/2}\\ &\quad\times(16S_n^*((x^{-4\alpha}+t^{-4\alpha}),x))^{1/2}\\ &\quad+(x^{2-\alpha}/(1-\alpha))^2\,e^{-nx}\\ &\leqslant 4M_4(x/n)^2\,(x^{-4\alpha}+M_{4\alpha}x^{-4\alpha})^{1/2}\\ &\quad+2(1-\alpha)^{-2}\,x^{2-2\alpha}(nx)^2\,e^{-nx}/2!\,n^{-2}\\ &\leqslant M_5x^{2-2\alpha}n^{-2},\end{aligned}$$

where M_4 and M_5 are constants independent of x and n.

Thus, for $x > 1/n$, we also have

$$\begin{aligned}w(x)\,|S_n(f_h,x)-f_h(x)| &\leqslant 9M_f'' h^{2\alpha-2}(1+x)^{-b}\,M_b(1+x)^b\sqrt{M_5}\,x^{1-\alpha}n^{-1}\\ &\leqslant 9M_f'' M_6\sqrt{M_5}\,n^{-\alpha}.\end{aligned}$$

Therefore, we have

$$\sup_{x\geqslant 0}\{w(x)\,|S_n(f_h,x)-f_h(x)|\} = O(n^{-\alpha}).$$

From (4.5) we have

$$\begin{aligned} w(x)\,|S_n^*(f_h-f,x)| &\leqslant w(x)\,S_n^*\left((2/h)^2 \iint_0^{h/2} M_f'' t^{-a-\alpha}\right.\\ &\qquad \left.\times (1+t+2u+2v)^b\,(u+v)^{2\alpha}\,du\,dv,\,x\right)\\ &\leqslant w(x)\,M_f'' h^{2\alpha} S_n^*(t^{-a-\alpha}(1+t+2h)^b,x)\\ &\leqslant M_f'' w(x)\,h^{2\alpha}(S_n^*(t^{-2a-2\alpha},x))^{1/2}\\ &\qquad \times (2^{2b} S_n^*((2h)^{2b}+(1+t)^{2b},x))^{1/2}\\ &\leqslant M_f'' w(x)\,h^{2\alpha} M_{a,\alpha} x^{-a-\alpha} 2^b((2h)^{2b}+M_b(1+x)^{2b})^{1/2}\\ &\leqslant M_f'' M_{a,\alpha} 2^b (4^b+M_b)^{1/2}\,n^{-\alpha}. \end{aligned}$$

Note that $f \in C^0_{a,b}$. We also have

$$\begin{aligned} w(x)\,|S_n(f_h-f,x)| &\leqslant w(x)\,|S_n^*(f_h-f,x)| + w(x)(2/h)^2\\ &\qquad \times \iint_0^{h/2} |\Delta^2_{u+v} f(0)|\,du\,dv\,p_{n,0}(x)\\ &\leqslant w(x)\,|S_n^*(f_h-f,x)| + w(x)(2/h)^2 M_f''\\ &\qquad \times \iint_0^{h/2} (u+v)^{\alpha-a}\,(1+u+v)^b\,du\,dv\,p_{n,0}(x). \end{aligned}$$

For the second term we have

$$\begin{aligned} &w(x)(2/h)^2 M_f'' \iint_0^{h/2} (u+v)^{\alpha-a}\,(1+u+v)^b\,du\,dv\,p_{n,0}(x)\\ &\quad \leqslant M_f'' w(x)(2/h)^{2a}\,h^{\alpha}(1+h)^b \left(\iint_0^{h/2} (u+v)^{-1}\,du\,dv\right)^a p_{n,0}(x)\\ &\quad \leqslant M_f'' 2^{2a} x^a (1+x)^{-b}\,h^{\alpha-2a} 2^b (1+x)^b\,M_6^a (h/2)^{2a}\,h^{-a}\,p_{n,0}(x)\\ &\quad \leqslant 2^b M_f'' M_6^a (nx)^{(a+\alpha)/2}\,e^{-nx} n^{-\alpha}\\ &\quad \leqslant 2^b M_f'' M_6^a (2nxe^{-2nx/(a+\alpha)}/(a+\alpha))^{(a+\alpha)/2}\,(a+\alpha)^{(a+\alpha)/2}\,n^{-\alpha}\\ &\quad \leqslant 2^b M_f'' M_6^a (a+\alpha)^{(a+\alpha)/2}\,n^{-\alpha}, \end{aligned}$$

here we have used the following inequality in [1]

$$\iint_0^t (x+u+v)^{-1}\,du\,dv \leqslant M_6 t^2 (x+2t)^{-1} \qquad (0<t\leqslant 1), \tag{4.8}$$

and for $t \in [2^m, 2^{m+1})$, $m \in N$,

$$\iint_0^t (u+v)^{-1}\, du\, dv \leqslant \sum_{i,j=-\infty}^{m} \int_{2^i}^{2^{i+1}} \int_{2^j}^{2^{j+1}} (u+v)^{-1}\, du\, dv$$

$$\leqslant \sum_{i=-\infty}^{m} 2^i = 2^{m+1} \leqslant 4t^2/(2t),$$

hence (4.8) holds for $x=0$ and any $t>0$.

Finally, by (4.5) we have for $x>0$

$$w(x)\, |f(x)-f_h(x)| \leqslant w(x)(2/h)^2 \iint_0^{h/2} |\Delta^2_{u+v} f(x)|\, du\, dv$$

$$\leqslant w(x)(2/h)^2 \iint_0^{h/2} M_f'' x^{-a-\alpha}$$

$$\times (1+x+2u+2v)^b\, (u+v)^{2\alpha}\, du\, dv$$

$$\leqslant M_f''(2/h)^2\, (1+x)^{-b}\, (1+x+2h)^b\, x^{-\alpha} h^{2\alpha} (h/2)^2$$

$$\leqslant 3^b M_f'' n^{-\alpha}.$$

Combining all the above discussions we obtain

$$w(x)\, |S_n(f, x) - f(x)| \leqslant M_f n^{-\alpha},$$

where M_f is a constant independent of n and x.

The proof of our main result is complete.

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